

Remark on quasi-ideals of ordered semigroups

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Abstract

The aim is to correct part of the Remark 3 of my paper “On regular, intra-regular ordered semigroups” in Pure Math. Appl. (P.U.M.A.) 4, no. 4 (1993), 447–461. On this occasion, some further results and the similarity between the *po*-semigroups and the *le*-semigroups is discussed.

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1 Introduction and prerequisites

There is a typing mistake in Remark 3 in [8], corrected by hand in the reprints: On page 458, line –4, the set $X \cup ((XS] \cap (SX])$ should be replaced by $\left(X \cup ((XS] \cap (SX]) \right]$. In this Remark we have seen that, for the subset $Q := X \cup ((XS] \cap (SX])$ of S , we have

1) $(QS] \cap (SQ] \subseteq Q$ and

2) if T is a quasi-ideal of S such that $X \subseteq T$, then $Q \subseteq T$, thus $(Q] \subseteq (T] = T$. In addition, we have $((Q]) = Q$. As a consequence, the set $(Q]$ is the quasi-ideal of S generated by X . Indeed:

$$((Q]S] \cap (S(Q]) = ((Q](S]) \cap ((S](Q]) \subseteq (QS] \cap (SQ] \subseteq Q.$$

Though the following does not affect the proof, on page 459, line 13, it is better to write $\left(X \cup ((XS] \cap (SX]) \right]$ instead of $\left(X \cup ((XS] \cap (SX]) \right)$; furthermore in line 16 of the same page X^2S should be replaced by $(X^2S]$ taking it into account in the rest of the proof of the Remark. For convenience, we give here detailed proofs of our arguments.

For an ordered semigroup S and a subset H of S , we denote by $(H]$ the subset of S defined by

$$(H) := \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

A nonempty subset A of S is called a quasi-ideal of S if (1) $(AS] \cap (SA] \subseteq A$ and (2) if $a \in A$ and $S \ni b \leq a$, then $b \in A$ (equivalently $(A] \subseteq A$, which in turn is equivalent to $(A] = A$). It is called a bi-ideal of S if (1) $ASA \subseteq A$ and (2) if $a \in A$ and $S \ni b \leq a$, then $b \in A$. Every quasi-ideal of S is a bi-ideal of S as well. Indeed, if B is a quasi-ideal of S , then $BSB \subseteq (BS] \cap (SB] \subseteq B$. A nonempty subset A of S is called a left (resp. right) ideal of S if (1) $SA \subseteq A$ (resp. $AS \subseteq A$) and (2) if $a \in A$ and $S \ni b \leq a$, then $b \in A$. For a subset A of S , the set $(A \cup AS]$ is the right ideal of S generated by A , and the set $(A \cup SA]$ is the left ideal of S generated by A . An ordered semigroup S is called intra-regular if for every $a \in S$ there exist $x, y \in S$ such that $a \leq xa^2y$. This is equivalent to saying that $a \in (Sa^2S]$ for every $a \in S$ or $A \subseteq (SA^2S]$ for every $A \subseteq S$ (cf., for example [7]).

We mention the properties we use in the paper: Clearly $S = (S]$, and for subsets A, B of S , we have the following: if $A \subseteq B$, then $(A] \subseteq (B]$; $A \subseteq (A]$; $((A]) = (A]$; $(A](B] \subseteq (AB]$; $((A](B]) = (AB]$ (cf., for example [6]).

2 Main results

If S is an ordered semigroup, for a subset X of S , the set

$$Q := \left(X \cup ((XS] \cap (SX]) \right]$$

is the quasi-ideal of S generated by X . It is mentioned without proof in [9]. In spite of whatever we already said at the beginning of the introduction which gives a complete proof of our argument, we think it is interesting to give an independent detailed proof which is the following:

1) $(QS] \cap (SQ] \subseteq Q$. Indeed:

$$\begin{aligned} QS &= \left(X \cup ((XS] \cap (SX]) \right] S = \left(X \cup ((XS] \cap (SX]) \right] (S] \\ &\subseteq \left(X \cup (XS] \right] (S] \subseteq \left((X \cup (XS]) S \right] \\ &= (XS \cup (XS]S]. \end{aligned}$$

Since $(XS]S = (XS][S] \subseteq (XS^2] \subseteq (XS]$, we have

$$QS \subseteq \left(XS \cup (XS] \right] = \left((XS] \right] = (XS],$$

so $(QS] \subseteq \left((XS] \right] = (XS]$. Similarly we get so $(SQ] \subseteq (SX]$, thus we have

$$\begin{aligned} (QS] \cap (SQ] &\subseteq (XS] \cap (SX] \subseteq X \cup \left((XS] \cap (SX] \right) \\ &\subseteq \left(X \cup \left((XS] \cap (SX] \right) \right] = Q. \end{aligned}$$

2) If $a \in Q$ and $S \ni b \leq a$, then $b \in Q$. In fact:

Since $a \in Q := \left(X \cup \left((XS] \cap (SX] \right) \right]$, we have

$$b \leq a \leq t \text{ for some } t \in X \cup \left((XS] \cap (SX] \right).$$

If $t \in X$, then $b \leq t \in X$, then $b \in (X] \subseteq \left(X \cup \left((XS] \cap (SX] \right) \right] = Q$.

If $t \in (XS] \cap (SX]$, then $b \leq t \in (XS] \cap (SX]$, so

$$b \in \left((XS] \cap (SX] \right] \subseteq \left(X \cup \left((XS] \cap (SX] \right) \right] = Q.$$

[Instead of 2), we could also write $((Q]) = (Q]$ (as this holds for any subset of S).

3) If T is a quasi-ideal of S such that $T \supseteq X$, then

$$Q := \left(X \cup \left((XS] \cap (SX] \right) \right] \subseteq \left(T \cup \left((TS] \cap (ST] \right) \right] = (T] = T.$$

□

The sufficient condition of Proposition 2 in [8] is the following:

Suppose $X \cap Q \cap Y \subseteq (YQX]$ for every right ideal X , every left ideal Y and every quasi-ideal Q of S . Then S is intra-regular.

Here is its corrected proof: Let $X \subseteq S$. Denote by $r(X)$, $l(X)$, $q(X)$ the right ideal, left ideal and the quasi-ideal of S , respectively, generated by X . By hypothesis, we have

$$\begin{aligned} X &\subseteq r(X) \cap q(X) \cap l(X) \subseteq \left(l(X)q(X)r(X) \right] \\ &= \left((X \cup SX] \left(X \cup \left((XS] \cap (SX] \right) \right) (X \cup XS] \right] \\ &= \left((X \cup SX) \left(X \cup \left((XS] \cap (SX] \right) \right) (X \cup XS) \right] \end{aligned}$$

$$\begin{aligned}
&\subseteq \left[(X \cup SX) \left(X \cup (XS] \right) (X \cup XS) \right] \\
&= \left[\left(X^2 \cup SX^2 \cup X(XS] \cup SX(XS] \right) (X \cup XS) \right]
\end{aligned}$$

Since $X(XS] \subseteq (X](XS] \subseteq (X^2S]$ and $SX(XS] \subseteq (SX](XS] \subseteq (SX^2S]$, we get

$$\begin{aligned}
X &\subseteq \left[\left(X^2 \cup SX^2 \cup (X^2S] \cup (SX^2S] \right) (X \cup XS) \right] \\
&= \left[X^3 \cup SX^3 \cup (X^2S]X \cup (SX^2S]X \cup X^3S \cup SX^3S \cup (X^2S]XS \cup (SX^2S]XS \right].
\end{aligned}$$

Since

$$\begin{aligned}
(X^2S]X &\subseteq (X^2S](X] \subseteq (X^2SX] \subseteq (X^2S], \\
(SX^2S]X &\subseteq (SX^2S](X] \subseteq (SX^2SX] \subseteq (SX^2S], \\
(X^2S]XS &\subseteq (X^2S](XS] \subseteq (X^2SXS] \subseteq (X^2S] \text{ and} \\
(SX^2S]XS &\subseteq (SX^2S](XS] \subseteq (SX^2SXS] \subseteq (SX^2S], \text{ we obtain}
\end{aligned}$$

$$X \subseteq \left[X^3 \cup SX^2S \cup (X^2S] \cup (SX^2S] \right] = \left[X^3 \cup (SX^2S] \cup (X^2S] \right].$$

Then

$$\begin{aligned}
X^3 &\subseteq \left[X^3 \cup (SX^2S] \cup (X^2S] \right] X^2 \subseteq \left[X^3 \cup (SX^2S] \cup (X^2S] \right] (X^2] \\
&\subseteq \left[\left(X^3 \cup (SX^2S] \cup (X^2S] \right) X^2 \right] \\
&= \left[X^5 \cup (SX^2S]X^2 \cup (X^2S]X^2 \right].
\end{aligned}$$

Since

$$\begin{aligned}
X^5 &\subseteq SX^2S \\
(SX^2S]X^2 &\subseteq (SX^2S](X^2] \subseteq (SX^2SX^2] \subseteq (SX^2S] \text{ and} \\
(X^2S]X^2 &\subseteq (X^2S](X^2] \subseteq (X^2SX^2] \subseteq (X^2S], \text{ we have}
\end{aligned}$$

$$X^3 \subseteq \left[SX^2S \cup (SX^2S] \cup (X^2S] \right] = \left[(SX^2S] \cup (X^2S] \right].$$

Then

$$\begin{aligned}
X &\subseteq \left[X^3 \cup (SX^2S] \cup (X^2S] \right] \\
&\subseteq \left[\left((SX^2S] \cup (X^2S] \right) \cup (SX^2S] \cup (X^2S] \right] \\
&= \left[\left((SX^2S] \cup (X^2S] \right) \right] \\
&= \left[(SX^2S] \cup (X^2S] \right],
\end{aligned}$$

and hence

$$\begin{aligned}
X^2 &\subseteq X((SX^2S] \cup (X^2S]) \subseteq (X)((SX^2S] \cup (X^2S]) \\
&\subseteq \left(X((SX^2S] \cup (X^2S]) \right) = (X(SX^2S] \cup X(X^2S]) \\
&\subseteq ((X)(SX^2S] \cup (X)(X^2S]) \subseteq ((XSX^2S] \cup (X^3S]) \\
&\subseteq ((SX^2S]) = (SX^2S],
\end{aligned}$$

$$\begin{aligned}
X^2S &\subseteq (SX^2S]S = (SX^2S](S] \subseteq (SX^2S^2] \subseteq (SX^2S] \text{ and} \\
(X^2S] &\subseteq ((SX^2S]) = (SX^2S].
\end{aligned}$$

Thus we have

$$X \subseteq ((SX^2S] \cup (X^2S]) = ((SX^2S]) = (SX^2S],$$

and S is intra-regular. \square

Combining this result with the Proposition 2 in [8], we get the following theorem:

Theorem 1. *Let S be an ordered semigroup. The following are equivalent:*

- (1) S is intra-regular;
- (2) For every right ideal X , every left ideal Y and every bi-ideal B of S , we have $X \cap B \cap Y \subseteq (YBX]$;
- (3) For every right ideal X , every left ideal Y and every quasi-ideal Q of S , we have $X \cap Q \cap Y \subseteq (YQX]$.

As we already know, the theory of ordered semigroups based on ideals and the theory of le -semigroups based on ideal elements are parallel to each other. All the results on le -semigroups based on ideal elements are expressed in ordered semigroups in terms of ideals, and conversely. It is surprising that, for the results on ordered semigroups based on ideals points do not play any essential role but the sets [8], as we have also seen in the results above. In this respect, the Theorem 1, in case of le -semigroups is the following:

Theorem 2. (cf. also [5]) *Let S be an le -semigroup. The following are equivalent:*

- (1) S is intra-regular;
- (2) For every right ideal element x , every left ideal element y and every bi-ideal element b of S , we have $x \wedge b \wedge y \leq ybx$;

- (3) For every right ideal element x , every left ideal element y and every quasi-ideal element q of S , we have $x \wedge q \wedge y \leq yqx$.

Let us first give the necessary definitions, and then we will prove the theorem. A *poe*-semigroup is an ordered semigroup (*po*-semigroup) S having a greatest element usually denoted by “ e ” (that is, $e \geq a$ for all $a \in S$) [4]. An *le*-semigroup is a *poe*-semigroup which is at the same time a lattice (under the operations \vee and \wedge) such that $a(b \vee c) = ab \vee ac$ and $(a \vee b)c = ac \vee bc$ for all $a, b, c \in S$ [1, 2]. An element a of an ordered semigroup S is called a right (resp. left) ideal element if $ax \leq a$ (resp. $xa \leq a$) for all $x \in S$ [1]. If S is a *poe*-semigroup, then a is a right (resp. left) ideal element of S if and only if $ae \leq a$ (resp. $ea \leq a$) [4]. An element a of a *poe*-semigroup S is called a bi-ideal element of S if $aea \leq a$, and it is called a quasi-ideal element of S if the element $ae \wedge ea$ exists (in S) and $ae \wedge ea \leq a$ [3]. For an *le*-semigroup, we denote by $r(a)$, $l(a)$, $q(a)$ the right ideal element, the left ideal element and the quasi-ideal element of S , respectively, generated by a . We have $r(a) = a \vee ae$, $l(a) = a \vee ea$. A *poe*-semigroup S is called intra-regular if $a \leq ea^2e$ for all $a \in S$ (cf., for example [4]). The element $a \vee (ae \wedge ea)$ is the quasi-ideal element of S generated by a ($a \in S$). In fact:

$$(a \vee (ae \wedge ea))e \wedge e(a \vee (ae \wedge ea)) = (ae \vee (ae \wedge ea)e) \wedge (ea \vee e(ae \wedge ea)).$$

Since $ae \wedge ea \leq ae$, we have $(ae \wedge ea)e \leq ae^2 \leq ae$. Since $ae \wedge ea \leq ea$, we have $e(ae \wedge ea) \leq e^2a \leq ea$. Thus we have

$$(a \vee (ae \wedge ea))e \wedge e(a \vee (ae \wedge ea)) = ae \wedge ea \leq a \vee (ae \wedge ea).$$

Clearly, $a \vee (ae \wedge ea) \geq a$. If t is a quasi-ideal element of S such that $t \geq a$, then $a \vee (ae \wedge ea) \leq t \vee (te \wedge et) \leq t$.

Proof of the theorem. (1) \implies (2). If S is intra-regular, then for any element a of S , we have

$$a \leq ea^2e = eaae \leq e(ea^2e)(ea^2e)e \leq ea^2ea^2e.$$

Let now x be a right ideal element, y a left ideal element and b a bi-ideal element of S . Then, for the element $x \wedge b \wedge y$ of S , we have

$$\begin{aligned} x \wedge b \wedge y &\leq e(x \wedge b \wedge y)(x \wedge b \wedge y)e(x \wedge b \wedge y)(x \wedge b \wedge y)e \\ &\leq (ey)(beb)(xe) \leq ybx. \end{aligned}$$

(2) \implies (3). This is because the quasi-ideal elements of S are bi-ideal elements of S as well. Indeed, if q is a quasi-ideal element of S , then $qeq \leq qe \wedge eq \leq q$.

(3) \implies (1). Let $a \in S$. Then $a \leq ea^2e$. In fact: By hypothesis, we have

$$\begin{aligned}
a &\leq r(a) \wedge q(a) \wedge l(a) \leq l(a)q(a)r(a) \\
&= (a \vee ea) \left(a \vee (ae \wedge ea) \right) (a \vee ae) \\
&\leq (a \vee ea)(a \vee ea)(a \vee ae) \\
&= (a^2 \vee ea^2 \vee aea \vee eaea)(a \vee ae) \\
&= a^3 \vee ea^3 \vee aea^2 \vee eaea^2 \vee a^3e \vee ea^3e \vee aea^2e \vee eaea^2e \\
&\leq a^3 \vee ea^2e \vee ea^2
\end{aligned}$$

Then

$$a^3 \leq (a^3 \vee ea^2e \vee ea^2)a^2 = a^5 \vee ea^2ea^2 \vee ea^4 \leq ea^2e.$$

Thus we have

$$a \leq ea^2e \vee ea^2, \quad a^2 \leq ea^2ea \vee ea^3 \leq ea^2e, \quad ea^2 \leq ea^2e.$$

Thus we get $a \leq ea^2e$, and S is intra-regular. \square

Remark. The implication (1) \Rightarrow (2) of Theorem 2 holds in a *poe*-semigroup S in general for which for any right ideal element x , every left ideal element y and every bi-ideal element b of S , the element $x \wedge b \wedge y$ exists.

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